Relativistic Quantum Mechanics :-

Klein - Gordon

It is the combination of (Quantum mechanics + special theory of relativity). This theory is highly accurate. In this theory the motion of that particle is considered whose speed is comparable to the speed of light.

Klein - Gordon eq^n :-

This eq^n is also known as relativistic wave eq^n and it is equivalent to schroedinger's wave eq^n. It is second order differentiation eq^n in space & time, satisfying the Lorentz transformation. This eq^n may be used for (non - relatistic) as well as fast (relativistic). It is only appreciable for spinless particle. The wave func^n of K-G eq^n is not same as obtained. This eq^n does not contain potential term.

Schroedinger eq^n is given by

$$\hat{H}\psi = i\hbar \frac{\partial \psi}{\partial t} \quad \text{——— (1)}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2 \nabla^2}{2m} \quad \text{——— (2)}$$

Substituting in (1), we get

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(r,t) = i\hbar \frac{\partial \psi}{\partial t}(r,t) \quad \text{——— (3)}$$

This eq^n is invariant under Gallelian transformations or Linear Transformations but it is not invariant under Lorentz transformation in Special theory of relativity. Therefore, this eq^n is not applicable in relativistic systems.

In relativity, the energy i.e; Hamiltonian 'H' for a free particle is given by:-

$$\hat{H} = E = \pm \sqrt{(\hat{p}^2 c^2 + m^2 c^4)}$$

$$\hat{H} = E = \pm [\hat{p}^2 c^2 + m^2 c^4]^{1/2} \quad \text{——— (4)}$$

The Schroedinger eq^n becomes :-

$$\pm\left[\hat{p}^2 c^2 + m^2 c^4\right]^{1/2}\psi = i\hbar \frac{\partial \psi}{\partial t} \quad (5)$$

This is Schroedinger eq^n in relativistic medium but this eq^n could not satisfy the energy states as obtained by Bohr and Sommerfield. So, Schroedinger was disappointed and put aside this eq^n. After this, Klein & Gordon modified this eq^n. They found the following difficulties in this eq^n :- (7)

1. The first difficulty is of '+ve' and '-ve' signs. In classical mechanics, this was not problem because +ve and -ve energies, both exists and therefore the -ve energies can be ignored. But in quantum mechanics '-ve' energy does not exists.

2. It is not possible to interpret the square root of an operator i.e, the operator can not exists in its square root

form.

3. This eq^n is only applicable for spinless particle.

These above difficulties were removed approximately by Klein-Gordon. They operated the entire eq^n by 'H'.

From ①

$$\hat{H}\psi = i\hbar \frac{\partial \psi}{\partial t}$$

$$\hat{H}^2\psi = \hat{H}\left(i\hbar\frac{\partial\psi}{\partial t}\right)$$

$$= i\hbar \frac{\partial(\hat{H}\psi)}{\partial t}$$

$$= i\hbar\frac{\partial}{\partial t}\left(i\hbar\frac{\partial\psi}{\partial t}\right)$$

$$\hat{H}^2\psi = -\hbar^2\frac{\partial^2\psi}{\partial t^2} \quad \text{—— ⑥}$$

This is K-G eq^n or relativistic Sch. eq^n

where $E^2 = H^2$

For a free particle $\hat{H}^2 = \hat{p}^2c^2 + m^2c^4$

$$= \left(\frac{\hbar}{i}\nabla\right)^2 c^2 + m^2c^4$$

$$\hat{H}^2 = -\hbar^2 c^2 \nabla^2 + m^2 c^4 \quad \text{—— (7)}$$

put in (6)

$$\left(-\hbar^2 c^2 \nabla^2 + m^2 c^4\right)\psi = -\hbar^2 \frac{\partial^2 \psi}{\partial t^2}$$

$$\boxed{\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2 c^2}{\hbar^2}\right]\psi = 0} \quad \text{—— (8)}$$

This is Klein - Gordon eq^n for a free particle

or

Relativistic Schroedinger eq^n

or

Wave eq^n for relativistic systems.

This eq^n contain $\hbar$ (quantum mechanics) and c (relativity) i.e; It reflects quantum mechanics as well as relativity.

It may be written as :-

$$\boxed{\left[\Box^2 - \frac{m^2 c^2}{\hbar^2}\right]\psi = 0}$$

where $\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\nu}$

$\Box^2$ is known as D'Alembertian operator.

In co-ordinate representation equation (8) of K.G. Equation is

$$\left(g^{\mu\nu}\frac{\partial}{\partial x^\mu}\cdot\frac{\partial}{\partial x^\nu} - k^2\right)\psi(x^\mu) = 0.$$

Where $k = \frac{mc}{\hbar}$, in this form the invariance of K.G. equation is obvious

$$\nabla^2 - \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2}$$

space time

$$\frac{\partial}{\partial x^2} - \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2}$$ i.e; $x^2 - c^2 t^2$

Same as Lorentz transformation.